SYSTEM ANALYSIS
DATA, SYSTEMS, MODELLING

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1. Learning Objectives

☐ Scientific theories deal with concepts, not with reality.

☐ System theory uses mathematical concepts to describe aspects of the ‘real-world’.

☐ A formal model is a graph, i.e a subset of a product space formed by variables characterising a system or process.

☐ An observable is some characteristic of a system which can, in principle, be measured.

☐ A state is a specification of a system or process at a specific instant.

☐ A dynamic system or process is a system in which the state changes with time.

☐ Differential equations are a common way to encode dynamics.

☐ There are many alternative and equally valid ways to represent a system...
2. System Analysis

▷ The Modelling Relation.
▷ Natural System: physical, biological, financial, social,...
▷ Variables: \( x_1 \in X_1, x_2 \in X_2, x_3 \in X_3 \) taking values from
  \( X_1 = \{1, 2, 3, 4\}, \, X_2 = \{1, 2, 3, 4\}, \, X_3 = \{2, 8, 18, 32\} \)
▷ Data: \( M = \{m_j\}, \, j = 1, 2, \ldots, \, d \), where
  \( m_1 = (1, 1, 1), \, m_2 = (2, 2, 8), \, m_3 = (3, 3, 18), \, m_4 = (4, 4, 32) \).
▷ Model: \( x_3^2 = x_1^2 + x_2^2, \, x_3 = f(x_1, x_2) \), or equivalently
  \[
  F = \left\{ (x_1, x_2, x_3) : \, x_3 = f(x_1, x_2) \right\}
  \]
2.1. Modelling

1. Select variables (attributes).
2. Specify range of variables.
3. Sample, measure data.
4. Identify model $F$: Induction.
5. Describe constructive formulation $f(\cdot)$.

▷ Properties: Deduction, Simulation.
▷ Interpretation: Phenomenology, Semantics.
▷ Application: Decision Making.
How do we *encode* a system into a *formal model*...?
2.2. Observables

Proposition 1: “The only meaningful physical events which occur in the world are represented by the evaluation of observables on states.”

Proposition 2: “Every observable can be regarded as a mapping from states to real numbers.”

Definition 1 (Observables). An observable of a system is some characteristic which can, in principle, be measured. It is defined as a mapping from state space $X$ to the set of real numbers:

$$\xi_j : X \rightarrow \mathbb{R} \quad j = 1, 2, \ldots, n + m + l.$$ 

Equation of state:

$$f_i (\xi_1, \ldots, \xi_{n+m+l}) = 0 \quad i = 1, 2, \ldots, m.$$  \hfill (1)
Example:

Let the system under consideration be a closed vessel containing an ideal gas. Take $X$ to be the positions and velocities of the molecules making up the gas, and define the three observables for properties of the gas:

\[ P(x) = \text{pressure when in state } x, \]
\[ V(x) = \text{volume when in state } x, \]
\[ T(x) = \text{temperature when in state } x. \]

Then the ideal gas law asserts the single equation of state:

\[ f(P, V, T) = 0 \quad \text{specifically} \quad f(p, v, t) = pv - t. \]
2.3. Uncertainty

An observable $\xi$ on $X$ induces an equivalence relation

$$E_\xi(x, x') = 1 \text{ if and only if } \xi(x) = \xi(x')$$

and hence equivalence classes $[x]_\xi$ for which elements in $X$ are indistinguishable w.r.t $\xi$

$$[x]_\xi = \{x' : \xi(x') = \xi(x)\}.$$ 

The set of equivalence classes on $X$ is called quotient set and is denoted by $X/E_\xi$. Therefore, what we actually observe is usually not $X$ but the set of reduced states

$$X/E_\xi = \{[x]_\xi\}.$$ 

The modelling process itself can be discussed in terms of the linkage between observables. See your lecture notes for more details.
2.4. Parameters, Inputs, Outputs

Observables whose values remain fixed for every state \( x \in X \) are called parameters, \( \xi_i(x) = \theta_i \). For \( l \) parameters we write \( i = n + m + 1, n + m + 2, \ldots, n + m + l \),

\[
f (\xi_1, \ldots, \xi_{n+m}; \theta_1, \theta_2, \ldots, \theta_l) = 0 .
\]

If in addition \( m \) observables \( \xi_{n+1}, \xi_{n+2}, \ldots, \xi_{n+m} \) are functions of the remaining observables \( \xi_1, \xi_2, \ldots, \xi_n \), we use the notation

\[
\begin{align*}
\mathbf{u} & \triangleq [\xi_1, \xi_2, \ldots, \xi_n] \\
\mathbf{y} & \triangleq [\xi_{n+1}, \xi_{n+2}, \ldots, \xi_{n+m}] \\
\mathbf{\theta} & \triangleq [\theta_{n+m+1}, \theta_{n+m+2}, \ldots, \theta_{n+m+l}]
\end{align*}
\]

and obtain for the equation of state,

\[
f (\mathbf{u}; \mathbf{\theta}) = \mathbf{y} .
\]

We may then interpret the independent observables \( \mathbf{u} \), as inputs to the system and dependent observables \( \mathbf{y} \) as the resulting outputs.
2.5. The Graph of a System

State equation (2) suggest a model where \( f(\cdot) \) is a mapping relating inputs \( u \) directly to the outputs \( y \) without considering ‘inner states’:

\[
\begin{align*}
  f : U & \to Y \\
  u & \mapsto y .
\end{align*}
\]

(3)

Then any specific model describes a graph \( F \) of the mapping which represents system \( S \):

\[
F \subset U \times Y
\]

(4)

Example:
Let \( f : X \to \mathbb{R} \) be defined by the set of ordered triples \( (u_1, u_2, f(u_1, u_2)) \) such that each triple is belonging to \( \mathbb{R}^3 \), forming a surface

\[
F = \{(u_1, u_2, y) \in \mathbb{R}^3 : y = f(u_1, u_2)\}
\]
\[ f(u_1, u_2) \in F \]

- \( u_1 \) and \( u_2 \) are input variables.
- \( F \) is a set in \( \mathbb{R}^3 \).
- The diagram shows the relationship between the inputs and the output in \( \mathbb{R}^3 \).
3. Dynamic Systems

A *dynamical system* is one which changes in time.

**Example:** Newton’s particle mechanics.

Newton’s second law *defines* the force $F$, acting on a mass point $m$, to be the rate of change of momentum $m \cdot v$:

$$ F = \frac{d(m \cdot v)}{dt} = m \frac{d^2x}{dt^2}. $$

where $v$ denotes velocity (rate of change of position). With parameter $a$,

$$ F(x, v) = -a \cdot x. $$

We obtain the *equation of motion*

$$ m \frac{d^2x}{dt^2} = -a \cdot x. $$
The equation of motion is solved for $x$ as an explicit function of time. Alternative formulation of two first order ODEs:

\[
\frac{dx}{dt} = v
\]
\[
\frac{d(m \cdot v)}{dt} = -a \cdot x.
\]

Knowing the displacement and moment at an instant of time suffices to specify the state of the system hence the positions and momenta are called state variables.

In general,

\[
\frac{dx_i}{dt} = f_i(x_1, \ldots, x_r).
\]
Let $\xi, \xi'$ be two observables providing measurements of the position $x$ and its derivative. The *phase space* of the system is:

![Phase space diagram](image-url)
Matrix formulation: Let \( \mathbf{x} = [x_1, \ldots, x_r]^T \in \mathbb{R}^r \) and write,

\[
\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} \quad \text{such that} \quad \dot{\mathbf{x}} = F\mathbf{x}
\]

where \( F \) is a \( r \times r \) matrix, \( F \in \mathbb{R}^{r \times r} \), with constant coefficients. Let

\[
f: \mathbb{R}^r \rightarrow \mathbb{R}^r
\]

\[
\mathbf{x} \mapsto f(\mathbf{x}) = F\mathbf{x}.
\]

That is, a vector \( \mathbf{x} = [x_1, \ldots, x_r]^T \in \mathbb{R}^r \) is mapped to a vector \( f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_r(\mathbf{x})) \in \mathbb{R}^r \) with

\[
f_i(\mathbf{x}) = \sum_{j=1}^{r} a_{ij}x_j,
\]

where \( a_{ij} \) are the elements of the \( i \)th row of matrix \( F \). Thus \( F \) is a representation of the mapping \( f \). The solution of (5), for all \( t \), is obtained by integrating (5). The result is a family of solution curves, called trajectories.
4. Summary

- Modelling:
  - Variables, data, formal models.
  - The modelling process itself, linkage.
  - Observables, equation of state.

- System Models:
  - Parameters, inputs, outputs.
  - A formal system is a map, graph.

- Dynamic Systems:
  - State-space representation.
5. Further Reading


Rosen’s Modelling Relation

Phenomenal World

Mathematical World

NATURAL SYSTEM

FORMAL SYSTEM

ambience

causal entailment

encoding

decoding

the self

inferential entailment

components, function

phenomena, organisation.

propositions, axioms,

production rules,
algorithms.