Certain Uncertainty
Foundations of Fuzzy Mathematics

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1. Data and Information

Definition 1 (Cartesian Product (Space)). Let (x, y) be an *ordered pair*, where $x \in X$ and $y \in Y$, the *Cartesian product* is defined as the set

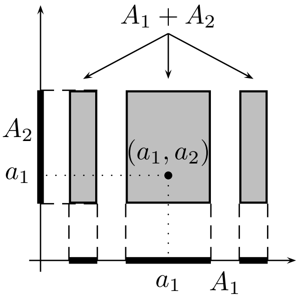
$$X \times Y = \{(x, y) \colon x \in X,\ y \in Y\} . \tag{1}$$

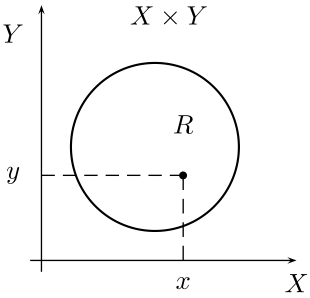
Definition 2 ((Binary) Relation). Any subset $R \subseteq X \times Y$ defines a (binary) *relation* between the elements of X and Y :

$$R = \{(x, y) \in X \times Y \colon R(x, y) \text{ holds}\} . \tag{2}$$

A relation is a *multi-valued* correspondence

$$\begin{aligned} R : \quad X \times Y &\to \{0, 1\} \\ (x, y) &\mapsto R(x, y) . \end{aligned}$$


$A_1 + A_2$
A_2
(a_1, a_2)
a_1
a_1
A_1


$X \times Y$
Y
R
y
x
X

2. Comparing

Definition 3 (Equivalence). Relation $R \subseteq X \times X$ establishes a relation among the elements of X. An *equivalence relation* on X is defined by the following conditions :

$$\begin{aligned} E(x,x) &= 1 \qquad \forall\, x \in X && \text{(reflexity)} \\ E(x,x') &= 1 \Rightarrow E(x',x) = 1 && \text{(symmetry)} \\ E(x,x') = 1 \;\wedge\; E(x',x'') &= 1 \Rightarrow \; E(x,x'') = 1 && \text{(transitivity)}\ . \end{aligned}$$

Example:

Consider the equivalence relation, called "equality" =.

$$\begin{aligned} a &= a \qquad \text{holds} && \text{(reflexity)} \\ a &= b \;\Rightarrow\; b = a && \text{(symmetry)} \\ a = b \;\wedge\; b &= c \;\Rightarrow\; a = c && \text{(transitivity)}\ . \end{aligned}$$

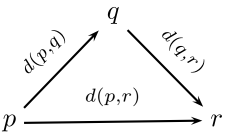
Definition 4 (Distance). The function $d(\cdot,\cdot)$ defines a *distance* between elements of X. Let $\forall\ p, q, r \in X$:

$$\begin{aligned} d(p,q) &= 0 && \text{iff} \quad p = q \\ d(p,q) &> 0 && \text{iff} \quad p \neq q \\ d(p,q) &= d(q,p) && \text{symmetry} . \end{aligned}$$

Definition 5 (Metric). A distance is called *metric* iff $\forall p, q, r \in X$ it is transitive :

$$d(p,r) \leq d(p,q) + d(q,r) \tag{3}$$

called *triangle inequality* :



Example:

$$d(x,x') = |x - x'| .$$

3. Order

Definition 6 (Partial Order). A *partial ordering* (or semi-ordering) on X is a binary relation $\preceq$ on X such that the relation is

reflexive, i.e. $x \preceq x$,

anti-symmetric, i.e. $x \preceq x'$ and $x' \preceq x$ implies $x = x'$,

transitive, i.e. $x \preceq x'$ and $x' \preceq x''$ implies $x \preceq x''$.

Example:

"greater or equal"	$\geq$:	$p > q$	$\wedge$	$q > r$	$\Rightarrow$	$p > r$
"set inclusion"	$\subseteq$:	$A \subseteq B$	$\wedge$	$B \subseteq C$	$\Rightarrow$	$A \subseteq C$

.. our ‘toolset’ so far:

Concept	Formalisation	Notation	
Data	product space	$X \times Y$	eq. (1)
Information	relations	$R(\cdot,\cdot)$	eq. (2)
Aggregation	union, intersection, compl.	$(\cup, \cap, ^c)$	-
Comparison	equivalence	$=$	df. (3)
	distance	$d(\cdot,\cdot)$	df. (4)
Order	$\preceq$	$\geq$, $\subseteq$	df. (6)
Reasoning	transitivity	triangle inequality	eq. (3)

These concept form a basic toolset for any scientific analysis of data and systems. But does it work?

4. Data Analysis

System Variables:	$p, q, r \in X$
Measurements:	$p = 1.5,\ q = 2,\ r = 2.2$
Error or Tolerance:	$\varepsilon = 0.6$
Model, Theory or Reasoning:	$p = q \ \wedge\ q = r \ \Rightarrow\ p = r$

...putting our theory into practise :

$$\begin{aligned} & |p-q| = 0.5 \ < \varepsilon \quad \Rightarrow \quad p = q \\ & |q-r| = 0.2 \ < \varepsilon \quad \Rightarrow \quad q = r \\ \text{but} \quad & |p-r| = 0.7 \ > \varepsilon \quad \Rightarrow \quad p \neq r \quad ✘ \end{aligned}$$

▷ called *The Poincaré paradox.*

Uncertainty is Certain!

5. Physical Reality vs. Mathematical Model

▷ *How can we preserve transitivity but take account of uncertainty?*

From the triangle inequality (3),

$$d(p,q) < a \quad \wedge \quad d(q,r) < b \;\Rightarrow\; d(p,r) < a + b \;.$$

Since $A \Rightarrow B$ implies that $Pr(A) \leq Pr(B)$, we get

$$Pr\big(d(p,q) < a \;\wedge\; d(q,r) < b\big) \leq Pr\big(d(p,r) < a + b\big)$$

Let $T(\cdot)$ be a function such that $T(Pr(A), Pr(B)) \leq Pr(A \wedge B)$ for any two propositions A, B. Then with $Pr\big(d(p,r) < a + b\big) = F_{pr}(a + b)$,

$$\boxed{F_{pr}(a+b) \geq T\left(F_{pq}(a), F_{qr}(b)\right)} \tag{4}$$

Certainty is lost – we have a choice for $T(\cdot)$.

6. Triangular Norms

The function $T(\cdot)$ [1] is a mapping $[0,1] \times [0,1] \rightarrow [0,1]$. For example,

$$T_{\text{min}}(a,b) = \min(a,b) \qquad \text{(minimum operator)},$$
$$T_{\text{Luk}}(a,b) = \max(a+b-1,0) \qquad \text{(Lukasiewicz norm)},$$
$$T_{\text{pro}}(a,b) = a \cdot b \qquad \text{(algebraic product)}.$$

Triangular or t-norms, so called because they preserve transitivity w.r.t the triangular inequality, play an important role in fuzzy mathematics.

Let $T(a,b) = a \cdot b$, then for (4),

$$F_{pr}(a+b) \geq F_{pq}(a) \cdot F_{qr}(b)$$

or in other words

$$Pr\big(d(p,r) < a+b\big) \geq Pr\big(d(p,q) < a,\ d(q,r) < b\big) \ .$$

7. Fuzzy Relations

▷ *Lotfi Zadeh [4] showed that any metric (in the unit interval) induces a similarity relation from the reference set into the unit interval:*

$$\widetilde{E}(x, x') = 1 - \inf\big(d(x, x'), 1\big) \tag{5}$$

If we are to interpret $\widetilde{E}(x, x')$ as the likelihood *that the distance between x and x' is zero or the* similarity *of x' w.r.t x, what is the formal definition of such a fuzzy equivalence relation called similarity relation?*

Example: Student Evaluation

Grade boundaries :

Degree	$\geq 50\%$
Distinction	$\geq 70\%$

Problem: Someone with, say 69.4%, may not achieve a distinction while a colleague with just 0.6% more would succeed.

Let (X, d) be a metric space with $X = [0, 100]$ and the metric

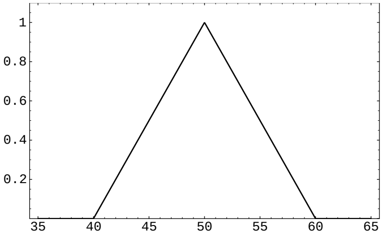
$$\begin{aligned} d\colon\ \mathbb{R} \times \mathbb{R} &\rightarrow \mathbb{R}^+ \\ (x, x') &\mapsto |x - x'| \ . \end{aligned}$$

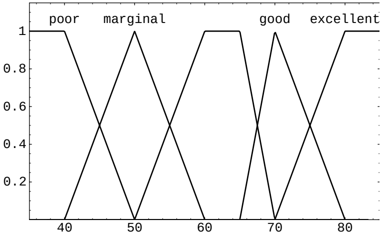
For any of the grade boundaries x_0, the induced *proximity relation* depends on x_0 and a 'locality' parameter $\delta \in (0, \infty)$:

$$\mu_{x_0}(x) = 1 - \min\{|\delta \cdot x_0 - \delta \cdot x|, 1\} \ . \tag{6}$$

Fuzzy Partions

Figure on the left: $\delta = 0.1$ and $x_0 = 50$.





Definition 7 (Similarity Relation). Any fuzzy subset S of the product space $X \times Y$ defines a (binary) *fuzzy relation* between the elements of X and Y.

$$\begin{aligned} \widetilde{E} : \quad X \times Y &\rightarrow [0,1] \\ (x,y) &\mapsto \widetilde{E}(x,y) \ . \end{aligned}$$

▷ *A similarity relation is reflexive* $S(x,x) = 1$ *and symmetric* $S(x,x') = S(x',x)$ *but is is an equivalence relation?*

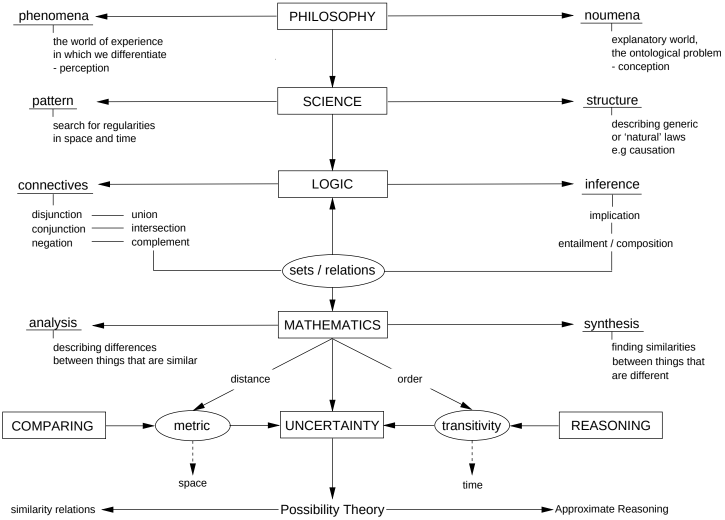
From Karl Menger's inequality [1],

$$F_{pr}(a+b) \geq T\left(F_{pq}(a), F_{qr}(b)\right) \tag{4}$$

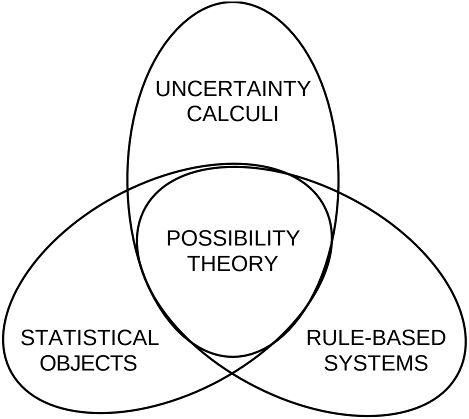
we obtain a definition of transitivity for similarity relations :

$$\boxed{\widetilde{E}(p,r) \geq T\left(\widetilde{E}(p,q), \widetilde{E}(q,r)\right)}$$

The Fuzzy Logic of Scientific Discovery



Possibility Theory



References

[1] Schweizer, B. and Sklar, A. : *Probabilistic Metric Spaces*. North-Holland, 1983. 11, 15

[2] Wolkenhauer, O. : *Possibility Theory with Applications to Data Analysis*. Research Studies Press, 1998.

[3] Wolkenhauer, O. : *Data Engineering: Data, Systems and Uncertainty*. Book manuscript, 1999. See http://www.csc.umist.ac.uk/people/wolkenhauer.htm.

[4] Zadeh, L.A. : *Similarity Relations and Fuzzy Orderings*. Information Sciences, Vol. 3, pp. 177–200, 1971. 12