System Analysis

Data, Systems, Modelling

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Contents

1. Learning Objectives

- ☐ Scientific theories deal with concepts, not with reality.
- ☐ System theory uses mathematical concepts to describe aspects of the 'real-world'.
- ☐ A formal model is a graph, i.e a subset of a product space formed by variables characterising a system or process.
- ☐ An observable is some characteristic of a system which can, in principle, be measured.
- ☐ A state is a specification of a system or process at a specific instant.
- ☐ A dynamic system or process is a system in which the state changes with time.
- ☐ Differential equations are a common way to encode dynamics.
- ☐ There are many alternative and equally valid ways to represent a system...

2. System Analysis

▷ The Modelling Relation.

▷ Natural System: physical, biological, financial, social,...

▷ Variables: $x_1 \in X_1$, $x_2 \in X_2$, $x_3 \in X_3$ taking values from

$$X_1 = \{1, 2, 3, 4\},\ X_2 = \{1, 2, 3, 4\},\ X_3 = \{2, 8, 18, 32\}$$

▷ Data: $\mathbf{M} = \{\mathbf{m}_j\}$, $j = 1, 2, \ldots, d$, where

$$\mathbf{m}_1 = (1, 1, 1),\ \mathbf{m}_2 = (2, 2, 8),\ \mathbf{m}_3 = (3, 3, 18),\ \mathbf{m}_4 = (4, 4, 32).$$

▷ Model: $x_3^2 = x_1^2 + x_2^2$, $x_3 = f(x_1, x_2)$, or equivalently

$$F = \left\{(x_1, x_2, x_3)\ :\ x_3 = f(x_1, x_2)\right\}$$

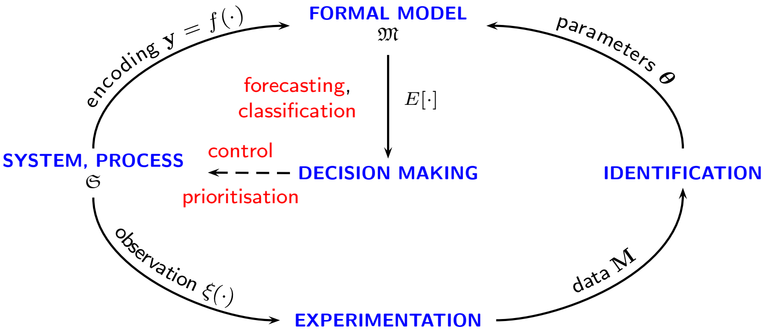
2.1. Modelling

1. Select variables (attributes).
2. Specify range of variables.
3. Sample, measure data.
4. Identify model F: *Induction*.
5. Describe constructive formulation $f(\cdot)$.

▷ Properties: *Deduction*, Simulation.

▷ Interpretation: Phenomenology, Semantics.

▷ Application: Decision Making.



How do we *encode* a system into a *formal model*...?

2.2. Observables

Proposition 1: "The only meaningful physical events which occur in the world are represented by the evaluation of observables on states."

Proposition 2: "Every observable can be regarded as a mapping from states to real numbers."

Definition 1 (Observables). An *observable* of a system is some characteristic which can, in principle, be measured. It is defined as a mapping from state space X to the set of real numbers :

$$\xi_j \colon X \to \mathbb{R} \qquad j = 1, 2, \dots, n+m+l \ .$$

Equation of state :

$$f_i\left(\xi_1, \dots, \xi_{n+m+l}\right) = 0 \qquad i = 1, 2, \dots, m \ . \tag{1}$$

Example:

Let the system under consideration be a closed vessel containing an ideal gas. Take X to be the positions and velocities of the molecules making up the gas, and define the three observables for properties of the gas

$$\begin{aligned} P(x) &= \text{pressure when in state } x, \\ V(x) &= \text{volume when in state } x, \\ T(x) &= \text{temperature when in state } x\ . \end{aligned}$$

Then the ideal gas law asserts the single equation of state

$$f(P, V, T) = 0 \qquad \text{specifically} \qquad f(p, v, t) = pv - t\ .$$

2.3. Uncertainty

An observable ξ on X induces an *equivalence relation*

$$E_\xi(x, x') = 1 \quad \text{if and only if} \quad \xi(x) = \xi(x')$$

and hence *equivalence classes* $[x]_\xi$ for which elements in X are indistinguishable w.r.t ξ :

$$[x]_\xi = \left\{x' : \ \xi(x') = \xi(x)\right\} .$$

The set of equivalence classes on X is called *quotient set* and is denoted by X/E_ξ. Therefore, what we actually observe is usually not X but the set of *reduced states*

$$X/E_\xi = \left\{[x]_\xi\right\} .$$

The modelling process itself can be discussed in terms of the *linkage* between observables. See your lecture notes for more details.

2.4. Parameters, Inputs, Outputs

Observables whose values remain fixed for every state $x \in X$ are called *parameters*, $\xi_i(x) = \theta_i$. For l parameters we write $i = n+m+1, n+m+2, \ldots, n+m+l$,

$$f\left(\xi_1, \ldots, \xi_{n+m};\ \theta_1, \theta_2, \ldots, \theta_l\right) = 0\ .$$

If in addition m observables $\xi_{n+1}, \xi_{n+2}, \ldots, \xi_{n+m}$ are functions of the remaining observables $\xi_1, \xi_2, \ldots, \xi_n$, we use the notation

$$\begin{aligned}\mathbf{u} &\doteq [\xi_1, \xi_2, \ldots, \xi_n] \\ \mathbf{y} &\doteq [\xi_{n+1}, \xi_{n+2}, \ldots, \xi_{n+m}] \\ \boldsymbol{\theta} &\doteq [\theta_{n+m+1}, \theta_{n+m+2}, \ldots, \theta_{n+m+l}]\end{aligned}$$

and obtain for the equation of state,

$$f(\mathbf{u}; \boldsymbol{\theta}) = \mathbf{y}\ . \tag{2}$$

We may then interpret the independent observables $\mathbf{u}$, as *inputs* to the system and dependent observables $\mathbf{y}$ as the resulting *outputs*.

2.5. The Graph of a System

State equation (2) suggest a model where $f(\cdot)$ is a mapping relating inputs $\mathbf{u}$ directly to the outputs $\mathbf{y}$ without considering 'inner states' :

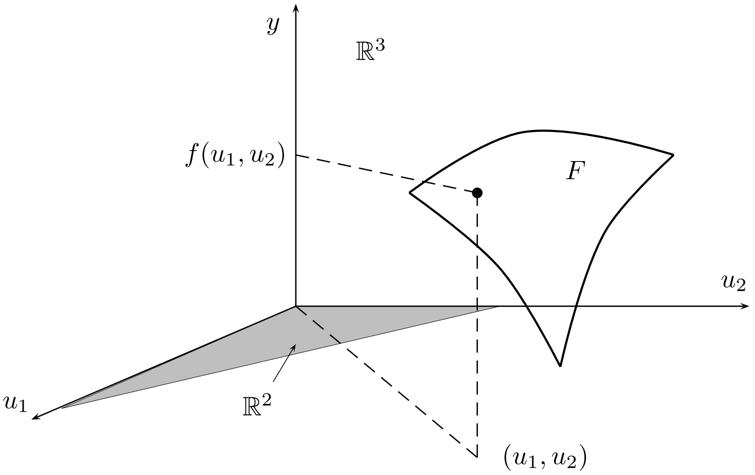
$$\begin{aligned} f \,:\, U &\rightarrow Y \\ \mathbf{u} &\mapsto \mathbf{y}\,. \end{aligned} \tag{3}$$

Then any specific model describes a *graph* F of the mapping which represents system $\mathfrak{S}$:

$$F \subset U \times Y \tag{4}$$

Example:
Let $f\colon X \rightarrow \mathbb{R}$ be defined by the set of ordered triples $\big(u_1, u_2, f(u_1, u_2)\big)$ such that each triple is belonging to $\mathbb{R}^3$, forming a surface

$$F = \big\{(u_1, u_2, y) \in \mathbb{R}^3\colon\, y = f(u_1, u_2)\big\}$$


y
$\mathbb{R}^3$
$f(u_1, u_2)$
F
u_2
u_1
$\mathbb{R}^2$
(u_1, u_2)

3. Dynamic Systems

A *dynamical system* is one which changes in time.

Example: Newton's particle mechanics.

Newton's second law *defines* the force F, acting on a mass point m, to be the rate of change of momentum $m \cdot v$:

$$F = \frac{\mathrm{d}(m \cdot v)}{\mathrm{d}t} \; = \; m \cdot \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} \; .$$

where v denotes velocity (rate of change of position). With parameter a,

$$F(x, v) = -a \cdot x \; .$$

We obtain the *equation of motion*

$$m \cdot \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -a \cdot x \; .$$

The equation of motion is solved for x as an explicit function of time. Alternative formulation of two first order ODEs :

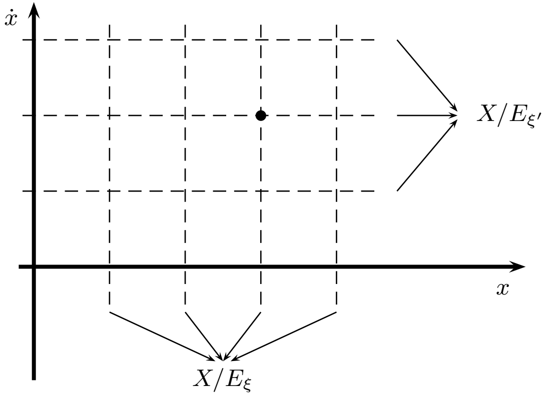
$$\frac{\mathrm{d}x}{\mathrm{d}t} = v$$
$$\frac{\mathrm{d}(m \cdot v)}{\mathrm{d}t} = -a \cdot x \ .$$

Knowing the displacement and moment at an instant of time suffices to specify the *state* of the system hence the positions and momenta are called *state variables*.

In general,

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = f_i\left(x_1, \ldots, x_r\right) \ .$$

Let ξ, ξ' bet two observables providing measurements of the position x and its derivative. The *phase space* of the system is :



Matrix formulation : Let $\mathbf{x} = [x_1, \dots, x_r]^T \in \mathbb{R}^r$ and write,

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} \doteq \dot{\mathbf{x}} \quad \text{such that} \quad \dot{\mathbf{x}} = \mathbf{F}\mathbf{x} \tag{5}$$

where $\mathbf{F}$ is a $r \times r$ matrix, $\mathbf{F} \in \mathbb{R}^{r \times r}$, with constant coefficients. Let

$$\begin{aligned} f: \ \mathbb{R}^r &\rightarrow \mathbb{R}^r \\ \mathbf{x} &\mapsto f(\mathbf{x}) = \mathbf{F}\mathbf{x} \ . \end{aligned}$$

That is, a vector $\mathbf{x} = [x_1, \dots, x_r]^T \in \mathbb{R}^r$ is mapped to a vector $f(\mathbf{x}) = \big(f_1(\mathbf{x}), \dots, f_r(\mathbf{x})\big) \in \mathbb{R}^r$ with

$$f_i(\mathbf{x}) = \sum_{j=1}^{r} a_{ij} x_j \ ,$$

where a_{ij} are the elements of the ith row of matrix $\mathbf{F}$. Thus $\mathbf{F}$ is a representation of the mapping f. The solution of (5), for all t, is obtained by integrating (5). The result is a family of solution curves, called *trajectories*.

4. Summary

- Modelling:
  - Variables, data, formal models.
  - The modelling process itself, linkage.
  - Observables, equation of state.
- System Models:
  - Parameters, inputs, outputs.
  - A formal system is a map, graph.
- Dynamic Systems:
  - State-space representation.

5. Further Reading

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3. Ljung, L. : *System Identification*. Prentice Hall, 1987.
4. Gershenfeld, N. : *The Nature of Mathematical Modelling*. Cambridge University Press, 1999.

Rosen's Modelling Relation