Fuzzy Mathematics

Fuzzy -Sets, -Relations, -Logic, -Graphs, -Mappings and The Extension Principle

Olaf Wolkenhauer

Control Systems Centre

UMIST

o.wolkenhauer@umist.ac.uk

www.csc.umist.ac.uk/people/wolkenhauer.htm

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1. Fuzzy Sets

Most commonly a *fuzzy set* is considered to be a family of pairs,

$$R = \{(x, \mu_R(x))\}$$

and *membership function* $\mu(\cdot)$.

Viewing degrees of membership as some kind of weighting on elements of the underlying reference space X, a *fuzzy restriction* is the mapping μ

$$\begin{aligned} \mu : \ X &\rightarrow [0, 1] \\ x &\mapsto \mu(x) \ . \end{aligned}$$

1.1. Soft/Hardware Implementations

For triangular and trapezoidal fuzzy set membership functions, let $a \leq b \leq c \leq d$ denote characteristic points :

"Left-open" set: $\mu(x; a, b) = \max\left(\min\left(\frac{b-x}{b-a}, 1\right), 0\right)$

"Right-open": $\mu(x; a, b) = \max\left(\min\left(\frac{x-a}{b-a}, 1\right), 0\right)$

"Triangular": $\mu(x; a, b, c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$

"Trapecoidal": $\mu(x; a, b, c, d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$

1.2. Level Set Representation

Instead of taking elements of the universe of discourse as arguments, we may consider the co-domain of μ to describe subsets of X in terms of α-cuts, R^α

$$R^\alpha = \{x : \mu_R(x) \geq \alpha\}$$

also called *level set*. Instead of taking a set-membership perspective we may view $\mu(\cdot)$ as a mapping, or fuzzy restriction R

$$\begin{aligned} \mu_R : X &\rightarrow L \\ x &\mapsto \alpha \end{aligned}$$

where here we assume $L = [0, 1]$. While in the set-membership setting we first identify a value x and then determine its degree of membership, we may also start with a level $\alpha \in L$ to find out which elements in X satisfy this condition.

The *representation theorem* shows that a family of sets $\{R^\alpha\}$, with the assertion "$\mathbf{x}$ is in R^α" has the "degree of truth" α, composes a fuzzy set or equivalently, fuzzy restriction :

$$R(x) = \sup_{\alpha\in(0,1]} \min\bigl(\alpha, \zeta_{R^\alpha}(\mathbf{x})\bigr)\ . \tag{1}$$

where

$$\alpha = \min\{\alpha, \zeta_{R^\alpha}(\mathbf{x})\}$$

and hence

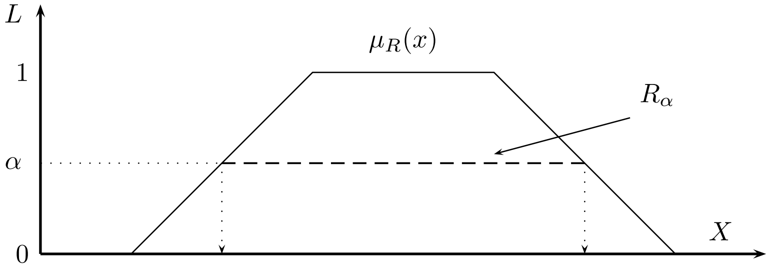
$$R = \Bigl\{(x,\alpha)\ :\ x\in X,\ \mu_R(x) = R(x) = \alpha\Bigr\}\ . \tag{2}$$

We can summarise the level-set representation of R as the mapping :

$$\begin{aligned} R\ :\ \bigl\{\langle R_\alpha\rangle\bigr\} &\ \rightarrow\ \mathcal{F}(X) \\ \langle R_\alpha\rangle &\ \mapsto\ \mu_R \end{aligned}$$

where

$$\mu_R(x) = \sup_{\alpha}\ \{\alpha\in[0,1],\ x\in R_\alpha\}\ .$$


L
$\mu_R(x)$
1
R_α
α
X
0

1.3. Algebra of Fuzzy Sets

For crisp sets, if X is any set and $x \in X$, the *algebra of the power set* $\mathcal{P}(X)$ of X, that is, of the set of (crisp) subsets of X, is usually formulated in terms of $A \in \mathcal{P}(X)$ and $B \in \mathcal{P}(X)$ as follows :

$$\begin{aligned} \text{Containment:} \quad & A \subset B \;\Leftrightarrow\; x \in A \;\Rightarrow\; x \in B \\ \text{Equality:} \quad & A = B \;\Leftrightarrow\; A \subset B \text{ and } B \subset A \\ \text{Complement:} \quad & A^c = \{x \in X \;:\; x \notin A\} \\ \text{Intersection:} \quad & A \cap B = \{x \in X \;:\; x \in A \text{ and } x \in B\} \\ \text{Union:} \quad & A \cup B = \{x \in X \;:\; x \in A \text{ or } x \in B \text{ or both}\} \end{aligned}$$

For fuzzy sets, since fuzzy sets are realised mathematically via functions, the set-theoretic operations and relations above have their equivalents in $\mathcal{F}(X)$, the set of all fuzzy sets (the set of all membership functions). Let μ_A, μ_B in $\mathcal{F}(X)$:

$$\begin{aligned}
\text{Containment:} &\quad A \subset B \;\Leftrightarrow\; \mu_A(x) \leq \mu_B(x) \\
\text{Equality:} &\quad A = B \;\Leftrightarrow\; \mu_A(x) = \mu_B(x) \\
\text{Complement:} &\quad \mu_{A^c}(x) = 1 - \mu_A(x) \\
\text{Intersection:} &\quad \mu_{A \cap B}(x) = T\big(\mu_A(x), \mu_B(x)\big) \\
\text{Union:} &\quad \mu_{A \cup B}(x) = S\big(\mu_A(x), \mu_B(x)\big)
\end{aligned}$$

✘ Note: These are by no means the only definitions but those most commonly used.

2. The Extension Principle

Instead of viewing the fuzzy systems as an algorithm based on formal multi-valued logic, the rule-base and inference mechanism can also be described as a mapping from a fuzzy set A' in X to a fuzzy set B' in Y.

In approximate reasoning, the compositional rule of inference generalised the 'crisp' rule

$$\text{IF } \mathbf{x} = a \text{ AND } \mathbf{y} = f(\mathbf{x}), \text{ THEN } y = f(a)$$

to be valid for fuzzy sets :

$$\mu_{B'}(y) = \sup_{x \in X} T\big(\mu_{A'}(x), \mu_R(x, y)\big) \ . \tag{3}$$

The fuzzy system, defined by the compositional rule of inference, maps fuzzy sets in X to fuzzy sets in Y. In other words, the fuzzy model describes a *fuzzy mapping*

$$\begin{array}{rcl} \tilde{f} : \mathcal{F}(X) & \to & \mathcal{F}(Y) \\ \mu_A(\mathbf{x}) & \mapsto & \widetilde{f}(A) \end{array} \tag{4}$$

where we obtain $\mu_{\tilde{f}(A)}(y)$ as a special case of the *composition of two fuzzy relations* :

$$\mu_{\tilde{f}(A)}(y) = \sup_{\mathbf{x}\in X} T\big(\mu_{A_{\mathrm{ext}}}(\mathbf{x}, y), \mu_R(\mathbf{x}, y)\big) \tag{5}$$

with extension $\mu_{A_{\mathrm{ext}}}(\mathbf{x}, y) = \mu_A(\mathbf{x})$, equivalent to (3) or the individual-rule based inference.

▷ We can take the extension of the mapping to the fuzzy mapping as a blueprint for a general *extension principle*.

▷ Let f be a mapping from X to Y, $y = f(x)$.

▷ Consider the situation where we are given a fuzzy number A ("approximately x_0") instead of a real number.

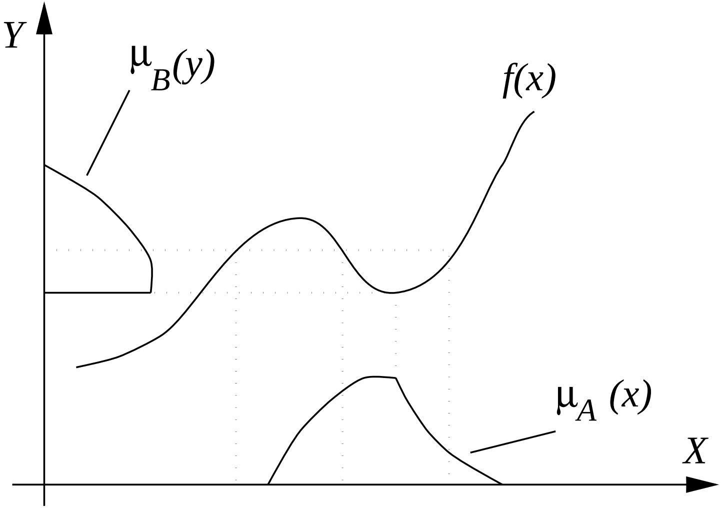
✘ We wish to find the fuzzy image B by a generalisation of f; how do we construct $B = f(A)$?

- ▷ We would require that the membership values of B should be determined by the membership values of A. Also $\sup B$ should be the image of $\sup A$ as defined by f.

- ▷ If the function f is surjective (onto), that is not injective (not a one-to-one mapping), we need to choose which of the values $\mu_A(x)$ to take for $\mu_B(y)$.

- ▷ Zadeh proposed the sup-union of all values x with $y = f(x)$ that have the membership degree $\mu_A(x)$. In other words,

$$\mu_B(y) = \sup_{x:\ y=f(x)} \mu_A(x) \ . \tag{6}$$



Y
μ_B(y)
f(x)
μ_A (x)
X

In general, we have the mapping

$$\begin{aligned} f \;:\; X_1 \times \cdots \times X_r &\quad\rightarrow\quad Y \\ (x_1, \ldots, x_r) &\quad\mapsto\quad y = f(x_1, \ldots, x_r) \end{aligned}$$

which we aim to generalise to a function $\tilde{f}(\cdot)$ of fuzzy sets. The extension principle is defined as

$$\begin{aligned} \tilde{f} \;:\; \mathcal{F}(X) &\quad\rightarrow\quad \mathcal{F}(Y) \\ \big(\mu_{A_1}(x_1), \ldots, \mu_{A_r}(x_r)\big) &\quad\mapsto\quad \tilde{f}\big(\mu_{A_1}(x_1), \ldots, \mu_{A_r}(x_r)\big) \;. \end{aligned}$$

where

$$\mu_{\tilde{f}(A_1,\ldots,A_r)}(y) = \sup_{(x_1,\ldots,x_r)\in f^{-1}(y)} \big\{\mu_{A_1}(x_1) \wedge \cdots \wedge \mu_{A_r}(x_r)\big\} \;. \qquad (7)$$

3. Fuzzy Graphs

There is a close relation between the concept of approximate reasoning, the fuzzy mapping introduced in the previous section and a *fuzzy graph* $\widetilde{F}$. Fuzzy rules and a fuzzy graph may both be interpreted as granular representations of functional dependencies and relations.

A fuzzy graph $\widetilde{F}$, serves as an approximate or compressed representation of a functional dependence $f\colon X \to Y$, in the form

$$\widetilde{F} = A_1 \times B_1 \ \vee A_2 \times B_2 \ \vee \cdots \vee \ A_{n_R} \times B_{n_R} \tag{8}$$

or more compactly

$$\widetilde{F} = \bigvee_{i=1}^{n_R} A_i \times B_i \ ,$$

where the A_i and B_i, $i = 1, \ldots, n_R$, are fuzzy subsets of X and Y, respectively; $A_i \times B_i$ is the cartesian product of A_i and B_i; and $\vee$ is the operation of disjunction, which is usually taken to be the union.

In terms of membership functions we may write

$$\mu_{\widetilde{F}}(x, y) = \bigvee_i \big(\mu_{A_i}(x) \wedge \mu_{B_i}(y)\big)$$

where $x \in X$, $y \in Y$, $\vee$ and $\wedge$ are any triangular t- and t-conorm, respectively. Frequently, $\vee = \max$ and $\wedge = \min$ establishing the relationship to the extension principle, approximate reasoning and so forth.

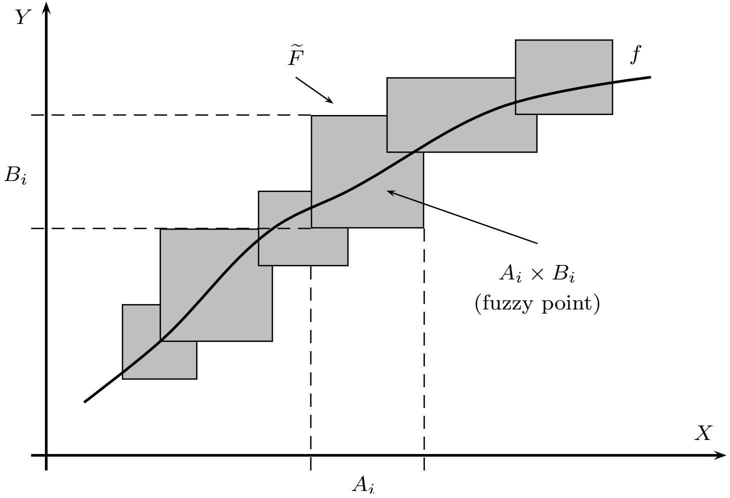
A fuzzy graph may therefore be represented as a fuzzy relation or a collection of fuzzy if-then rules

$$\text{IF } \mathbf{x} \text{ is } A_i, \text{ THEN } \mathbf{y} \text{ is } B_i \quad i = 1, 2, \ldots, n_R \ .$$

Each fuzzy if-then rule is interpreted as the joint constraint on $\mathbf{x}$ and $\mathbf{y}$ defined by

$$(\mathbf{x}, \mathbf{y}) \quad \text{is} \quad A_i \times B_i \ .$$

called a *fuzzy point* in $X \times Y$.


Y
$\widetilde{F}$
f
B_i
$A_i \times B_i$
(fuzzy point)
X
A_i

4. Fuzzy Logic

The basic claim of fuzzy theorists is that

Everything is a matter of degree!

The following paradoxes were first discussed by Bertrand Russel :

Liar paradox: Does the liar from Crete lie when he says that all Cretans are liars? If he lies, he tells the truth. But if he tells the truth, he lies.. .

Barber paradox: A barber advertises: “I shave all, and only, those men who don’t shave themselves”. Who shaves the barber? If he shaves himself, then according to the ad he doesn’t. If he does not, then according to him he does.. .

Set as a collection of objects:

i. Consider the collection of books. This collection itself is not a book, thus it is not a member of another collection of books.

ii. The collection of all things that are not books is itself not a book and therefore a member of itself.

iii. Now consider the set of all sets that are not members of themselves. Is this a member of itself or not?

- ▷ The problem is self-reference. The *law of the excluded middle* is violated:

$$t(S) \wedge t(\text{not } S) = 0 \qquad \text{or} \qquad S \cap S^c = \emptyset \; .$$

where t denotes the truth value (in $\{0, 1\}$), S is a statement or its set representation, respectively.

- ▷ In those paradoxes we find

$$t(S) = t(\text{not } S) \; . \tag{9}$$

- ▷ $t(\text{not } S) = 1 - t(S)$ inserted into (9) gives us the contradiction

$$t(S) = 1 - t(S) \; . \tag{10}$$

- ▷ However, in fuzzy logic we simply solve (10) for $t(S)$:

$$2 \cdot t(S) = 1 \qquad \text{or} \qquad t(S) = \frac{1}{2} \; .$$

.. the truth lies somewhere in between!