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ORIGINAL PAPER

Adjoints and emergence: applications of a new theory of adjoint functors

David Ellerman

Reviewed by

Olaf Wolkenhauer

www.sbi.uni-rostock.de

Ellerman’s goal is to provide an abstract analogy for the question:

“Given a domain of phenomena obeying certain laws, how can some qualitatively new and relatively autonomous behavior emerge?”

Content:

- Adjoint functors
- Adjunction (pair of adjoint functors)
- Heteromorphisms
- Determination through universals
  - Cartesian product
  - Coproduct
- Organism – Environment Relationships
- „Real-world“ examples

Category Theory

- Set theory is a biology of species. The categorical approach is a kind of sociology: One is no longer interested in the properties of the individual objects, but in their relationships. (Jet Nestruev)
- Morphisms express the transmission of determination between objects. (David Ellerman)
- The central structure is determination through ("self-participating, concrete") universals, expressed by Universal Mapping Properties (UMPs). (David Ellerman)
- "Adjoint functors are a tool to characterize what is important and universal in mathematics."
- Ellerman shows how adjunctions arise from the birepresentations of "heteromorphisms" between objects in different categories.

Universal Mapping Properties

- Universal mapping properties:
  - Initial object <> Terminal object
  - Sum of two objects <> Product of two objects
  - …
- Universal mapping properties come in pairs; the dual is obtained by reversing the maps.
- “The self-participating* universal for a property (if it exists) is the paradigmatic or archetypical example of the property.”
- I shall focus on Ellerman’s ideas to describe real-world systems with category theory.

* The universal “participates” in itself by the identity morphism.

Cone of Maps

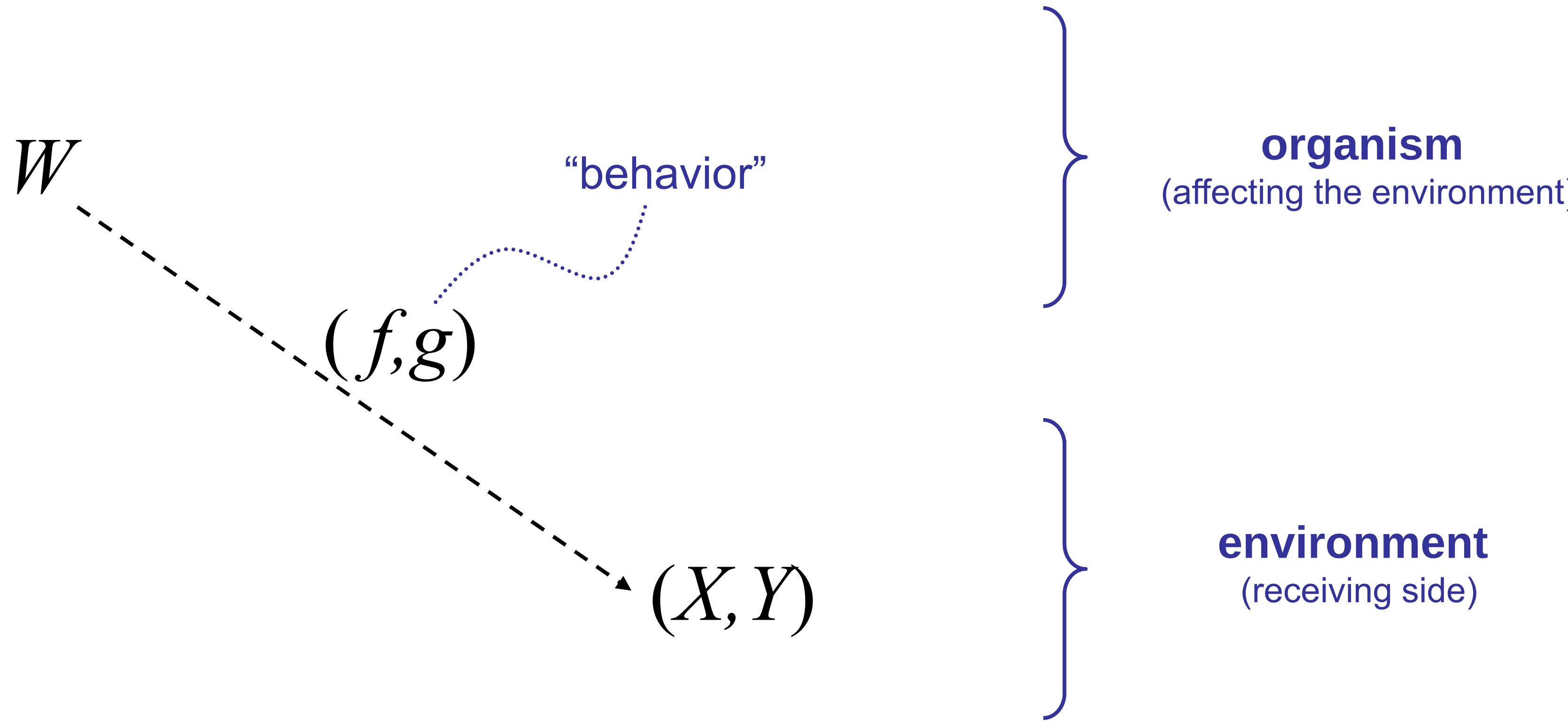
“determiners” (causes)

$$\begin{tikzcd}
 & X \\
W \arrow[ur, "f"] \arrow[dr, "g"'] & \\
 & Y
\end{tikzcd}
\qquad \Bigg\} \qquad
W \overset{(f,g)}{\dashrightarrow} (X, Y)$$

“cone of maps”

“determinees” (effects of determination)

Determination Through a Universal
(Case I: “sending through a universal”)



“To change this into determination through a universal, the „organism“ needs to internally construct a representation of the possible behaviors or external determinations.”

Determination Through a Universal

(Case I: "sending through a universal")

$$\begin{tikzcd} W \arrow[rr, "\langle f,g \rangle"] \arrow[drr, dashed, "(f,g)"'] & & X \times Y \\ & & (X,Y) \end{tikzcd}$$

"set of all possible effects"

"possible behaviors"

The Cartesian product is an internal representation of all possible determinations (f,g) in terms of all possible effects.

Determination Through a Universal
(Case I: “sending through a universal”)

$$
\begin{tikzcd}
W \arrow[rr, "\langle f,g \rangle"] \arrow[drr, dashed, "(f,g)"'] & & X \times Y \arrow[d, dashed, "(p_X,p_Y)"] \\
& & (X,Y)
\end{tikzcd}
$$

fixed

variable

canonical cone of maps

fixed

$$p_X((x,y)) = x$$

$$p_Y((x,y)) = y$$

The projection maps p_X, p_Y form a canonical cone between the representation $X \times Y$ of all the possible effects and the individual determinees or effects in X and in Y.

“The self-participating universal for a property (if it exists) is the paradigmatic or archetypical example of the property.”

$$
\begin{tikzcd}
W \arrow[rr, "\langle f,g \rangle"] \arrow[drr, dashed, "(f,g)"'] & & X \times Y \arrow[d, dashed, "(p_X,p_Y)"] \\
& & (X,Y)
\end{tikzcd}
$$

Property: Pair of maps (f,g) with common domain.
Self-participating Universal: Universal object $X \times Y$ and the projections (p_X, p_Y).

Given any other pair of (f,w), there is a unique factor map $h : W \to X \times Y$ such that $p_X h = f$ and $p_Y h = g$.

A pair (f,g) has the property iff it participates in (uniquely factors through) the self-participating universal (p_X, p_Y).

Determination Through a Universal

(Case I: “sending through a universal”)

object in the category of sets

$$\begin{tikzcd} W \arrow[r, "\langle f,g\rangle"] \arrow[dr, dashed, "(f,g)"'] & X \times Y \arrow[d, dashed, "(p_X,p_Y)"] \\ & (X,Y) \end{tikzcd}$$

product functor
(right-adjoint)

heteromorphism

object in the category of pairs of sets

Heteromorphism: morphism between objects in different categories.

Homomorphism: morphism between objects of the same category.

Determination Through a Universal

(Case I: “sending through a universal”)

“internal determination”

sending universal object

$$
\begin{tikzcd}
W \arrow[r, "\langle f,g \rangle"] \arrow[dr, dashed, "(f,g)"'] & X \times Y \arrow[d, dashed, "(p_X,p_Y)"] \\
& (X,Y)
\end{tikzcd}
$$

fixed

variable

sending universal morphisms

“external determination”

fixed

The internal universal model is successful if each external behavior (f,g) can be represented by a (unique) *internal* (factor) map $W \rightarrow X \times Y$ followed by a canonical projection.

The “universality” plus the “internality” will be later be combined to argue for some type of “autonomy” (addressing the issue of “emergence”).

Determination Through a Universal
(The receiving side to the sending universal)

$$
\begin{tikzcd}
W \arrow[rr, "\langle f,g \rangle"] \arrow[d, dashed, "(1_W,1_W)"'] \arrow[drr, dashed, "(f,g)"] & & X \times Y \arrow[d, dashed, "(p_X,p_Y)"] \\
\Delta W=(W,W) \arrow[rr, "(f,g)"'] & & (X,Y)
\end{tikzcd}
$$

The receiving universal is the dual concept, which together with the sending universal, forms a pair of adjoint functors.

For the receiving universal we reverse what is fixed and what is variable.

Adjunction

Adjunction: natural isomorphism between two sets of homomorphisms ("hom sets")

$$
\begin{tikzcd}
W \arrow[r, "\langle f,g \rangle"] \arrow[d, dashed, "(1_W,1_W)"'] \arrow[dr, dashed, "(f,g)"] & X \times Y \arrow[d, dashed, "(p_X,p_Y)"] \\
\Delta W=(W,W) \arrow[r, "(f,g)"'] & (X,Y)
\end{tikzcd}
$$

left-adjoint diagonal functor

right-adjoint product functor

$$Hom(\Delta W, (X,Y)) \approx Hom(W, X \times Y)$$

$$Hom(\Delta W, (X,Y)) \approx Het(W, (X,Y)) \approx Hom(W, X \times Y)$$

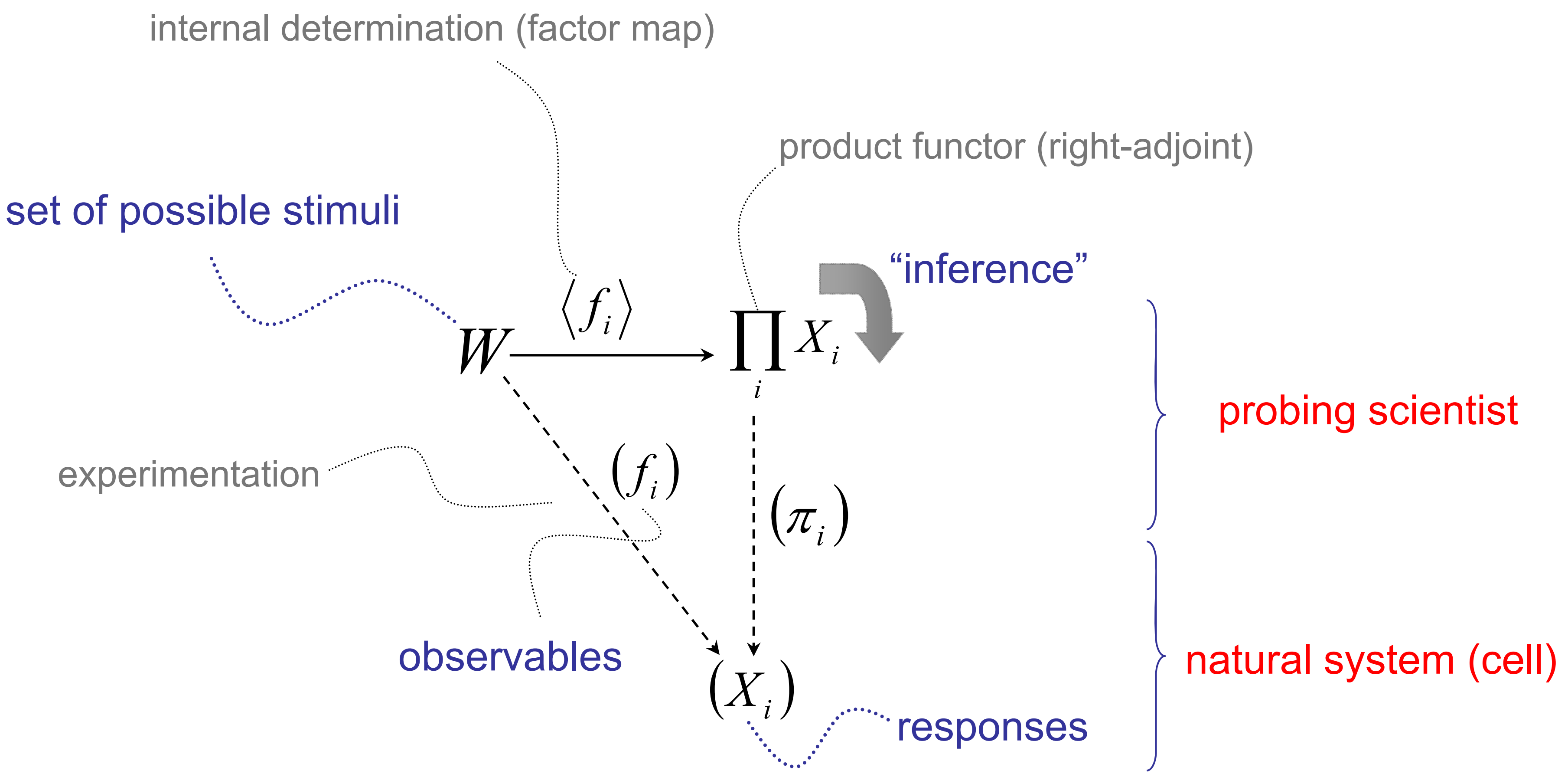
Determination Through a Universal
(Case I: "sending through a universal")

$$\begin{tikzcd}
W \arrow[rr, "\langle f,g \rangle"] \arrow[d, dashed, "{(1_W,1_W)}"'] \arrow[drr, dashed, "{(f,g)}"] & & X \times Y \arrow[d, dashed, "{(p_X,p_Y)}"] \\
\Delta W = (W,W) \arrow[rr, "{(f,g)}"'] & & (X,Y)
\end{tikzcd}$$

The universal construction of the product constructs the set of all determinees (or effects) so that the given instance of an *external* determination (f,g) factors through the universal by the *internal* map.

"The internal map "chooses" the effects and transmits the same results to (X,Y) as the original transmission from W to (X,Y)."

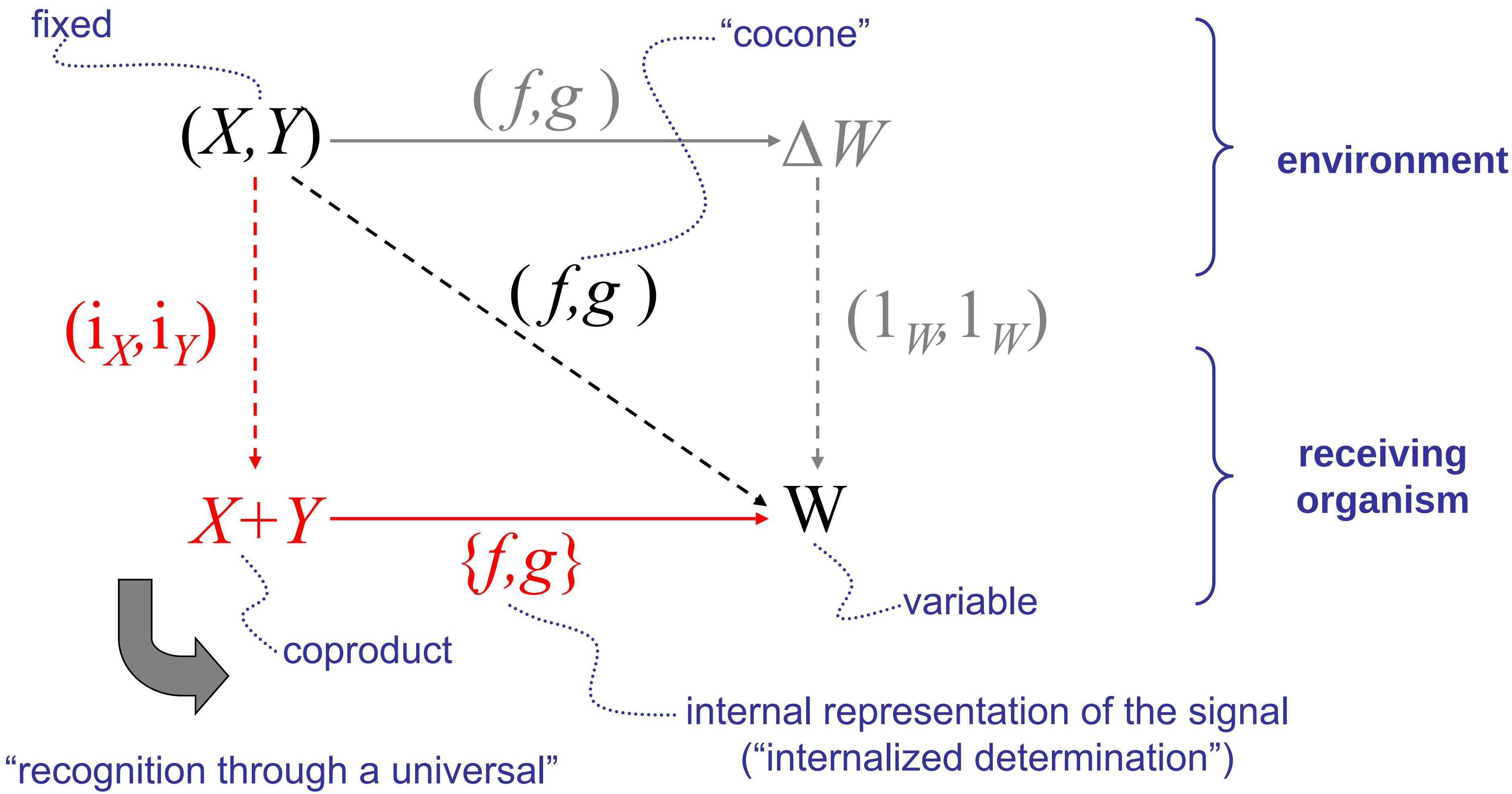
Modelling the modelling of biological cells



$$\forall W, (f_i): W \to (X_i) \quad \exists! \langle f_i \rangle : W \to \prod_i X_i \left[W \xrightarrow{\langle f_i \rangle} \prod_i X_i \xrightarrow{(\pi_i)} (X_i) = W \xrightarrow{(f_i)} (X_i) \right]$$

Determination Through a Universal

(Case II: "receiving through a universal")



Given any cocone (f,g) there is a unique factor map $\{f,g\}$ so that the internal reception of the signal through the receiving universal is the same as the original external signal.

Determination Through a Universal
(Case II: "receiving through a universal")

$$
\begin{tikzcd}
(X,Y) \arrow[r, "(f,g)"] \arrow[d, dashed, "(\mathrm{i}_X,\mathrm{i}_Y)"'] \arrow[dr, dashed, "(f,g)"] & \Delta W \arrow[d, dashed, "(1_W,1_W)"] \\
X+Y \arrow[r, "\{f,g\}"'] & \mathrm{W}
\end{tikzcd}
$$

diagonal functor, right-adjoint, assigning ΔW to W

coproduct functor, assigning $X+Y$ to (X,Y), left-adjoint

$$Hom(X+Y,\ W) \approx Het((X,Y),\ W) \approx Hom((X,Y), \Delta W)$$

Determination Through a Universal

(Case II: "receiving through a universal")

$$
\begin{tikzcd}
(X,Y) \arrow[rr, "(f{,}g)"] \arrow[d, dashed, "(i_X{,}i_Y)"'] \arrow[drr, dashed, "(f{,}g)"] & & \Delta W \arrow[d, dashed, "(1_W{,}1_W)"] \\
X+Y \arrow[rr, "\{f{,}g\}"'] & & W
\end{tikzcd}
$$

The universal construction of the coproduct constructs the set of all determiners or causes so that the given instance of an *external determination* factors through the universal by the *internal map* $\{f,g\}$.

"That internal map "recognizes" the causes and sends the same message to W as the original transmission from (X,Y) to W."

$$\begin{tikzcd}
(X,Y) \arrow[rr, "{(f,g)}"] \arrow[d, dashed, "{(i_X,i_Y)}"'] \arrow[drr, dashed, "{(f,g)}"] & & \Delta W \arrow[d, dashed, "{(1_W,1_W)}"] \\
X+Y \arrow[rr, "{\{f,g\}}"] & & W
\end{tikzcd}$$

Universality: While an external direct determination specifies or determines a particular set of possibilities, the determination through a universal constructs the object representing all the possibilities that might be directly determined—as indicated by its universal mapping property.

Autonomy: The universal is constructed in a manner independent of any external determiners (e.g., neither any x nor any f or g were involved in constructing the receiving universal

Self-determination: The morphism associated with the universal potentially determines all the possibilities

Indirectness: The particularization comes only with the indirect factor map that picks out or selects certain possibilities.

Composite Effect: The composition of the specific factor map followed by the universal morphism then implements the possibilities to agree with the given direct determination.

Cells as a selective/responsive system

signals (determiners)

all possible stimuli

$$
\begin{tikzcd}
(X_i) \arrow[d, dashed, "(\iota_i)"'] \arrow[dr, dashed, "(f_i)"] & \\
\coprod_i X_i \arrow[r, "\{f_i\}"] & W
\end{tikzcd}
$$

environment

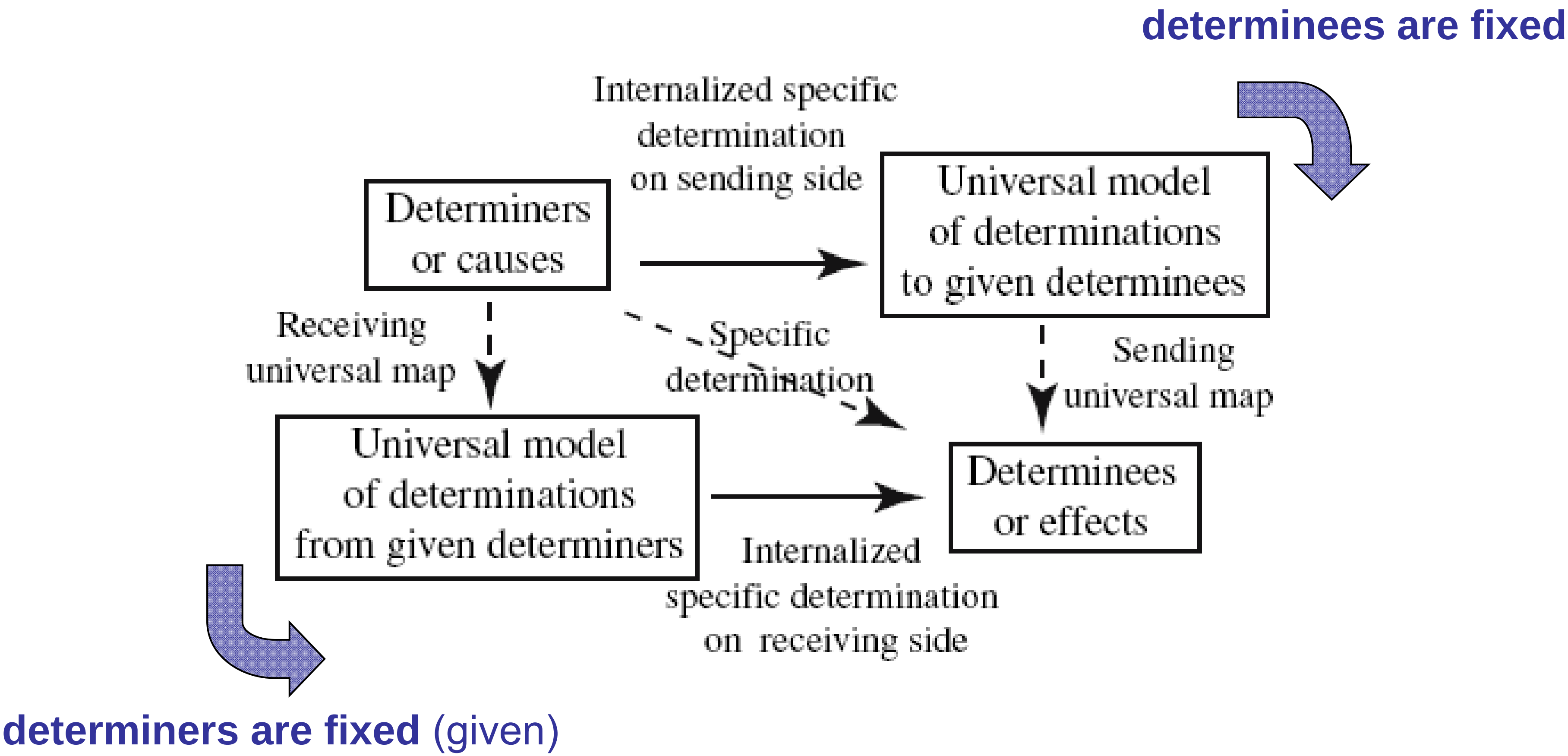
responding cell

“perception”, “recognition”

internalized determination

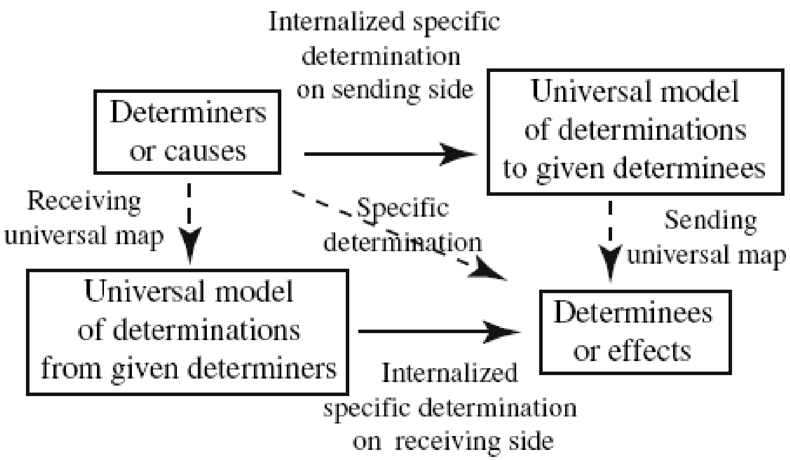
$$\forall (X_i),(f_i):(X_i)\rightarrow W \quad \exists!\{f_i\}:\coprod_i X_i \rightarrow W \left[(X_i)\xrightarrow{(\iota_i)}\coprod_i X_i \xrightarrow{\{f_i\}} W = (X_i)\xrightarrow{(f_i)} W \right]$$

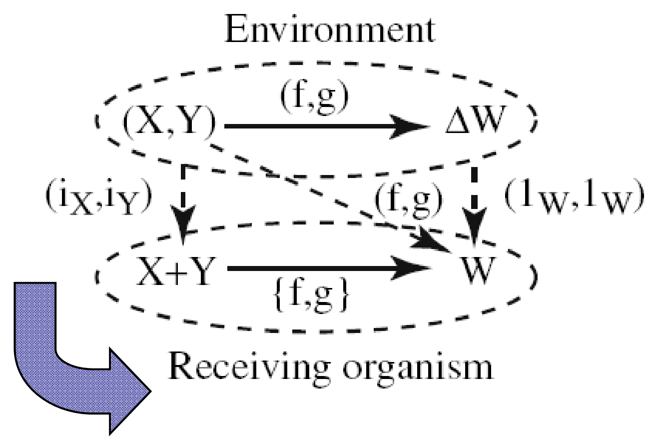
Summary: Determination through universals



Given a pair of adjoint functors (an adjunction),
there is always a sending universal and a receiving universal.

Summary: Determination through universals (Case II: “receiving through a universal”)





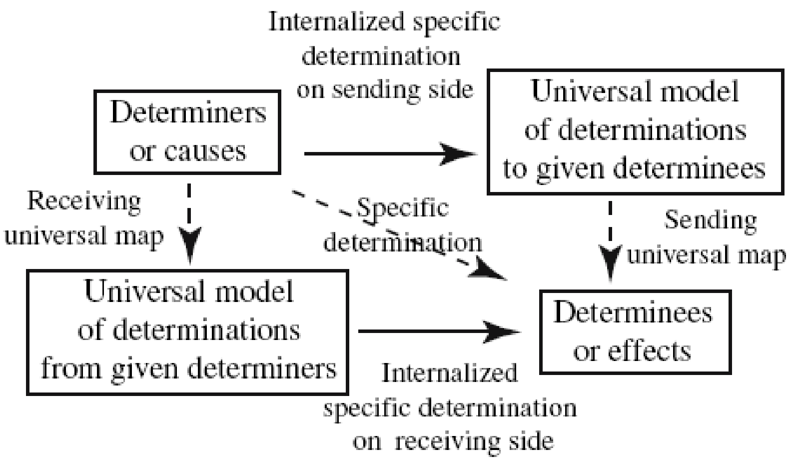
determiners are fixed (given)

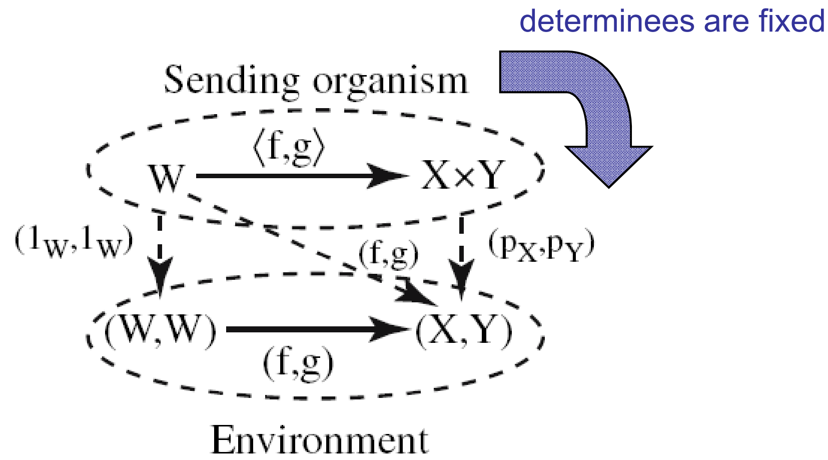
Universality While an external determination involves a given set of possible determiners, the determination through a universal constructs a universal object internal to the receiving side together with a universal receiving map so that all possible determinations from those determiners can be factored through that receiving universal.

Internalization The factorization through the universal internalizes the particular determination (e.g., (f, g) is replaced by $\{f, g\}$) so that the only external–internal connection is the indirect fixed canonical one connecting the external determiners to their internal representations (e.g., the canonical injections (i_X, i_Y) as the receiving universal map).

Autonomy = Universality + Internalization The net effect is that the receiver (‘‘organism’’) is ‘‘disconnected’’ from direct external stimulus control by the sender (the perception takes place, as it were, in the internalized ‘‘environment’’ or ‘‘world’’) and becomes in that sense autonomous.

Summary: Determination through universals (Case I: “sending through a universal”)





Universality While an external determination involves a given set of possible determinees or effects, the determination through a universal constructs a universal object internal to the sending side together with a universal sending map so that all possible determinations to those determinees can be factored through that sending universal.

Internalization The factorization through the universal internalizes the particular determination (e.g., (f, g) is replaced by $\langle f, g\rangle$) so that the only external–internal connection is the indirect fixed canonical one connecting the internal representations to the external effects (e.g., the canonical projections (p_X, p_Y) as the sending universal map).

Autonomy = Universality + Internalization The net effect is that the sender (‘‘organism’’) is ‘‘disconnected’’ from direct ‘‘causal’’ interaction with the effects (the action takes place, as it were, in the internalized ‘‘world’’) and becomes in that sense autonomous.

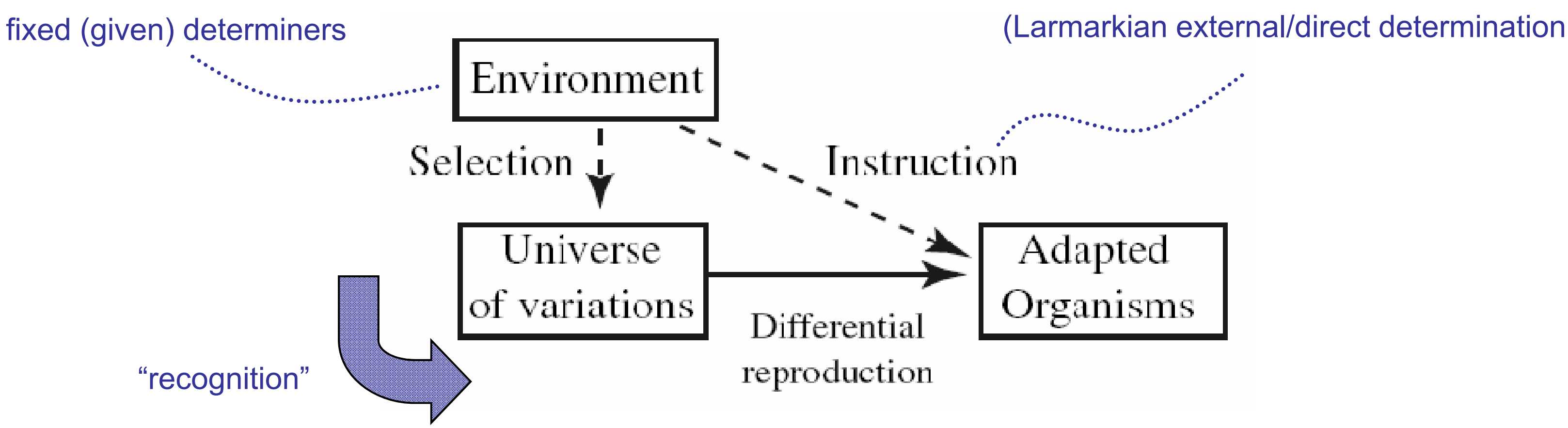
Emergence of Autonomous Behavior

"Given a domain of phenomena obeying certain laws, how can some qualitatively new and relatively autonomous behavior emerge?"

Ellerman gives several examples:

1. Selectionist vs. instructionist evolution.
2. The DNA mechanism as a universal constructor.
3. Selectionist versus instructionist theories of the immune system.
4. Edelman's selectionist theory of the brain.
5. Chomsky's theory of generative grammars.

Selectionist versus instructionist evolution

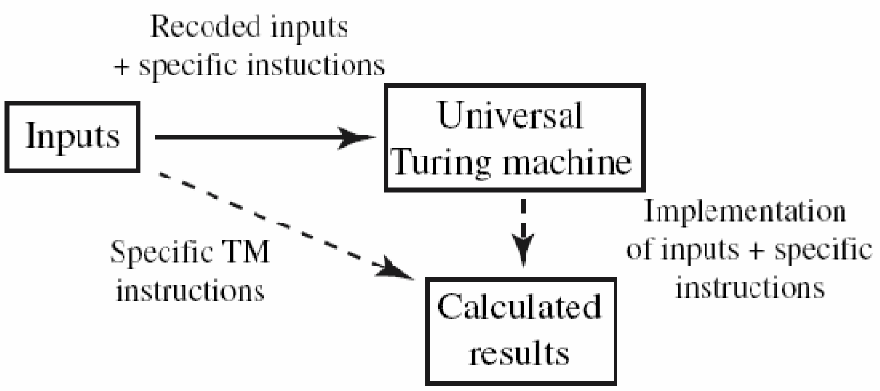


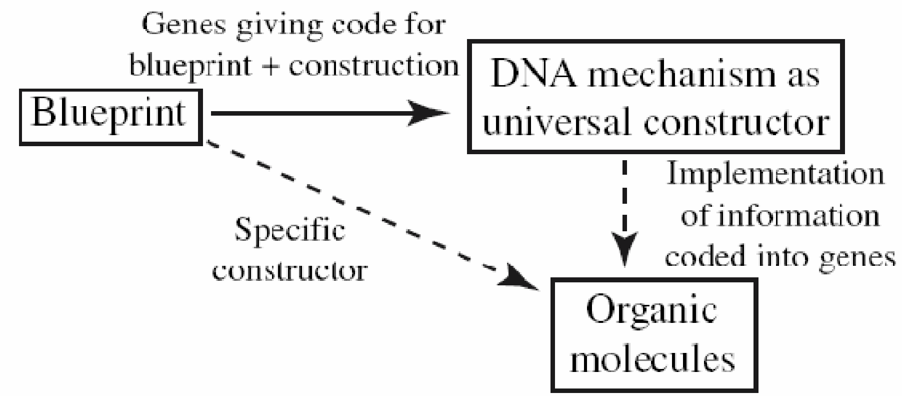
Universality The selectionist theory is an example of population thinking because it is the population, not the individual organism, that explores the universe of possibilities by variation through mutation and sexual reproduction.

Internalization The environment acts on the generated variety by selection and then, internal to the species, the fittest differentially reproduce so the net effect is "as if" the environment had directly instructed organisms with the fittest adaptations.

Autonomy In Darwinian theory, this is the basic non-Larmarckian point that there is no direct information flow from the environment to the organisms to somehow adapt certain characteristics. The actual process is the indirect one of generating a "universal" variety, and the environment selecting the fitter ones which then differentially reproduce.

The DNA mechanism as a universal constructor



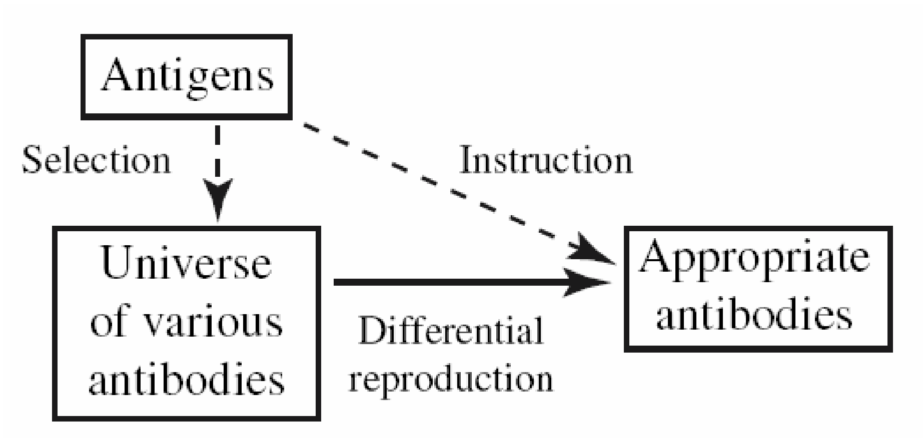


Universality As a sending universal, the DNA mechanism is structured to recognize and implement instructions for a given "universe" of relevant possible outcomes (amino acids, proteins, etc.).

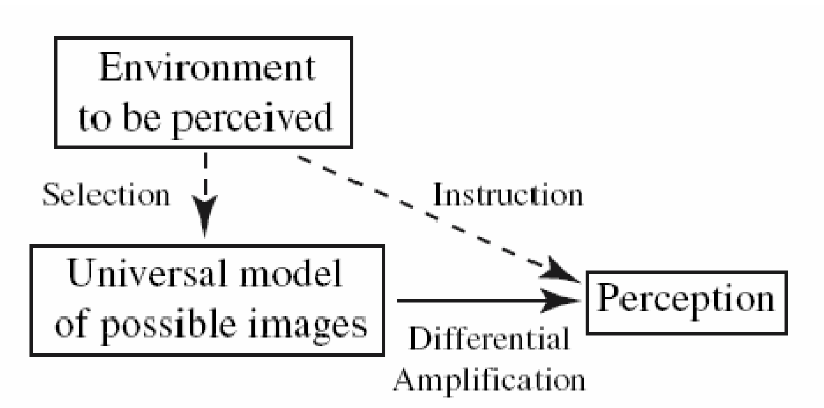
Internalization The genes plus the DNA mechanism combine to internalize one overall mechanism for the construction of the molecules.

Autonomy The net result of having the blueprint, specific construction instructions, and universal construction mechanism all internalized in a living organism gives a type of autonomy characteristic of living things.

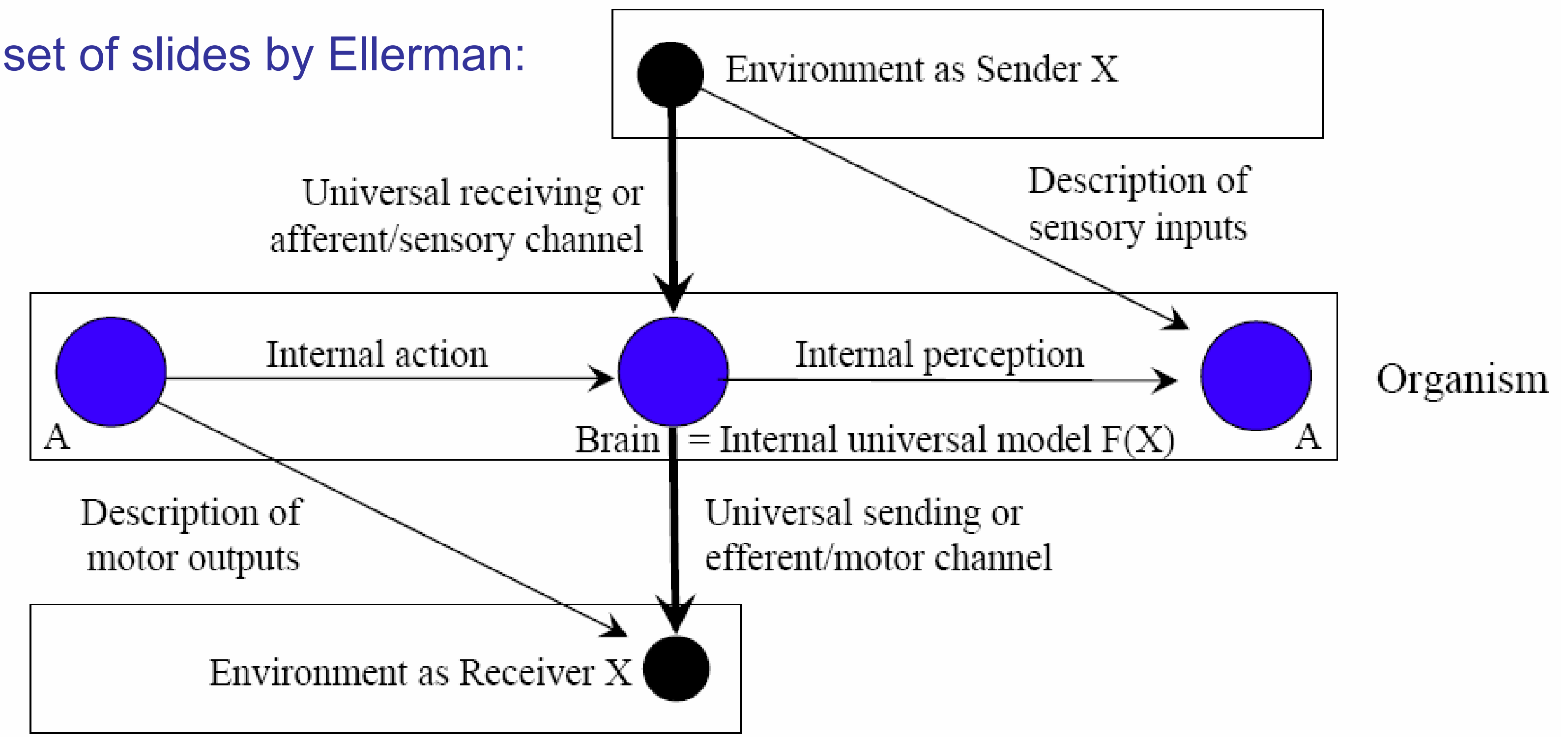
Selectionist versus instructionist theories of the immune system



Edelman’s selectionist theory of the brain



From another set of slides by Ellerman:

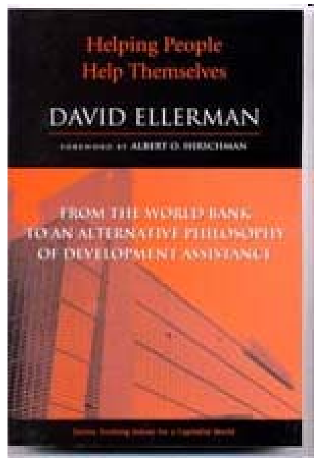


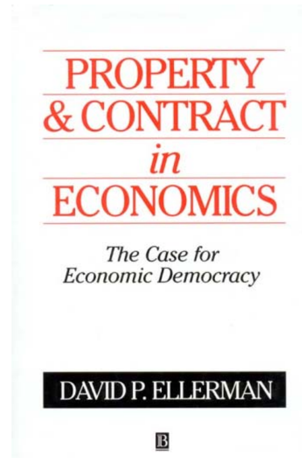
<u>Brain = functor giving left and right half-adjunction</u>

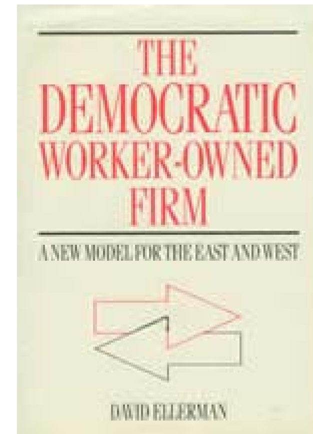
Further Reading

www.ellerman.org

David Ellerman







Amongst other things, there are more technical treatises of *Adjoint Functors and Heteromorphisms* on this website, including

A Theory of Adjoint Functors - with some Thoughts about their Philosophical Significance. In *What is Category Theory?* G. Sica (ed.), Polimetrica, Milan, 2006.